

DOMINATED SPLITTINGS AND THE SPECTRUM OF QUASI-PERIODIC JACOBI OPERATORS

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ABSTRACT. We prove that the resolvent set of any, possibly singular, quasi-periodic Jacobi operator is characterized as the set of all energies whose associated Jacobi cocycles induce a dominated splitting. This extends a well-known result by R. A. Johnson for Schrödinger operators.

1. INTRODUCTION

The purpose of this article is to give a dynamical characterization of the spectrum of quasi-periodic Jacobi operators (QPJ). To this end, let $\alpha \in \mathbb{R}^d$ be fixed with components linearly independent over \mathbb{Q} and let $c, v : \mathbb{T}^d \rightarrow \mathbb{C}$ be continuous satisfying $v(\mathbb{T}^d) \subseteq \mathbb{R}$. A *quasi-periodic Jacobi operator* is a family of bounded self-adjoint operators on $l^2(\mathbb{Z})$ of the form,

$$(1.1) \quad [H_x \psi]_n = \overline{c(\mathbb{T}^{n-1}x)} \psi_{n-1} + c(\mathbb{T}^n x) \psi_{n+1} + v(\mathbb{T}^n x) \psi_n, \text{ for } x \in \mathbb{T}^d,$$

generated upon evaluation of c, v along the trajectories of $T : \mathbb{T}^d \rightarrow \mathbb{T}^d, x \mapsto x + \alpha$. It is common to assume $\log |c| \in L^1(\mathbb{T}^d, d\mu)$, where μ is the Haar probability measure on \mathbb{T}^d . To simplify notation, we will set $X := \mathbb{T}^d$.

Motivated by the now famous almost Mathieu operator where for $d = 1$, $v(x) = 2\lambda \cos(2\pi x)$, $\lambda \in \mathbb{R}$, and $c \equiv 1$, most of the literature on QPJ has so far focussed on the special case where $c \equiv 1$, commonly known as quasi-periodic *Schrödinger operators* (QPS). In recent years, a dynamical systems approach to the spectral theory of the latter proved to be extremely fruitful. In particular, this approach allowed for a more global picture of the spectral properties of QPS [1, 2].

The relevant dynamical system for QPS are Schrödinger cocycles, quasi-periodic $SL_2(\mathbb{R})$ -cocycles whose iteration generate solutions to the finite difference equation, $H_x \psi = E\psi$. More generally, a (continuous) $M_2(\mathbb{C})$ -cocycle is a dynamical system on $X \times \mathbb{C}^2$ induced by T and a matrix-valued function $D \in \mathcal{C}(X, M_2(\mathbb{C}))$, defined by $(x, v) \mapsto (Tx, D(x)v)$. We will denote this cocycle map by the pair, (T, D) .

One fundamental ingredient of above-mentioned dynamical approach to the spectral theory of QPS is a theorem due to R. A. Johnson [19] which characterizes the spectrum in terms of Schrödinger cocycles. More specifically, it is shown in [19] that the resolvent set of a QPS is determined by uniformly hyperbolic dynamics of the Schrödinger cocycles. We mention that Johnson's theorem has recently

been generalized to discrete long-range Schrödinger operators [16]. The goal of this paper is to find an appropriate extension of Johnson's theorem to QPJ.

The main problem in extending Johnson's theorem to the more general Jacobi setting is that, whereas Schrödinger cocycles are $SL(2, \mathbb{R})$, the relevant cocycles for QPJ are in general not even invertible. In this context, we call a cocycle (T, D) *singular* if $\det D(x_0) = 0$ for some $x_0 \in X$. For QPJ, singular cocycles automatically arise once the sampling function c has zeros, in which case (1.1) is called a singular QPJ. We mention that QPJ originated in solid states physics, where both c, v are trigonometric polynomials, hence the possibility of c having zeros cannot be excluded in general. Even though several recent works on the spectral theory of QPJ have started to account for singularity [17, 29, 9, 10], an extension of Johnson's theorem to QPJ has so far been missing.

The presence of singular cocycles obviously requires a dynamical framework different from uniform hyperbolicity. Recent work on the continuity and positivity of the Lyapunov exponent for Jacobi operators [17, 9, 13, 14] indicated the notion “dominated splitting” as an appropriate analogue of uniform hyperbolicity, suitable when passing from the Schrödinger to the general Jacobi setting.

A cocycle (T, D) is said to induce a *dominated splitting* (write $(T, D) \in \mathcal{DS}$) if there exists $N \in \mathbb{N}$ and a continuous, non-trivial splitting of $\mathbb{C}^2 = S_x^{(1)} \oplus S_x^{(2)}$ satisfying $D_N(x)S_x^{(j)} \subseteq S_{T^N x}^{(j)}$, $1 \leq j \leq 2$, which exhibits uniform domination in the sense,

$$(1.2) \quad \frac{\|D_N(x)v_1\|}{\|v_1\|} > \frac{\|D_N(x)v_2\|}{\|v_2\|}, \text{ all } x \in X,$$

for all $v_j \in S_x^{(j)} \setminus \{0\}$. Here, for $n \in \mathbb{N}$, $D_n(x) := D(T^{n-1}x) \dots D(x)$ denotes the n -th iterate of (T, D) on the fibers, where $D_0(x) := I$. \mathcal{DS} is a very “robust property,” e.g. it is well known [23] that cocycles inducing a dominated splitting are open in $\mathcal{C}(X, M_2(\mathbb{C}))$, accompanied by continuity, even real analyticity of the (top) *Lyapunov exponent* (LE) ,

$$(1.3) \quad L(T, D) := \lim_{n \rightarrow +\infty} \frac{1}{n} \int \log \|D_n(x)\| d\mu(x) .$$

\mathcal{DS} specializes to uniform hyperbolicity (\mathcal{UH}) if the matrix cocycles are unimodular. More generally, for any *non-singular* cocycle, $(T, D) \in \mathcal{DS}$ if and only if $(T, \frac{D}{\sqrt{\det D}}) \in \mathcal{UH}$. The advantage of the notion \mathcal{DS} is however that it makes sense for *both* singular and non-singular cocycles.

Another feature not present for QPS, is that the relevant cocycles for QPJ (*Jacobi cocycles*) are not unique. For instance, one possible choice for Jacobi cocycles is given by

$$(1.4) \quad A^E(x) = \begin{pmatrix} E - v(x) & -\overline{c(T^{-1}x)} \\ c(x) & 0 \end{pmatrix} ,$$

where $E \in \mathbb{C}$ is the spectral parameter. There are however alternative choices, which, depending on the problem in mind, may be more advantageous. We emphasize that all these choices share that they are singular precisely if c has zeros. We will comment more on the flexibility in the choice of cocycles associated with the spectral theory of QPJ in Sec. 2. Our main result accounts for this flexibility, and gives a dynamical characterization of the spectrum of QPJ applicable for the different choices.

To formulate it, we recall that for a QPJ, minimality of T implies that the spectrum of the operators H_x is constant in $x \in X$; we denote this set by Σ , a compact subset of \mathbb{R} .

Theorem 1.1. *One has*

$$(1.5) \quad \Sigma = \{E : (T, A^E) \notin \mathcal{DS}\} .$$

Remark 1.2. (i) An analogous statement holds for the Jacobi cocycles alternative to (1.4) which will be described in Sec. 2, cf. Theorem 2.1.

(ii) Our proof of Theorem 1.1 does not depend on the specifics of the background dynamics induced by rotations on \mathbb{T}^d . Even though our main motivation for this work were QPJ, the argument we present applies to any uniquely ergodic map T on a compact Hausdorff space X where the T -invariant probability measure μ is topological, i.e. positive on open sets. In particular, Theorem 1.1 holds for all *almost periodic Jacobi operators*, i.e. operators of the form (1.1) where X is a compact topological group, μ is its Haar probability measure, and T is translation by a fixed element in X whose orbit is dense.

(iii) We mention that Johnson's original result extends to Schrödinger operators with background dynamics given by minimal transformation T on a compact space X . Our proof of Theorem 1.1 requires unique ergodicity of T (due to Proposition 4.1), we however believe that the result should also hold true for the case of merely minimal T .

From a dynamical point of view the most interesting aspect of Theorem 1.1 is that it implies domination outside the spectrum. In particular, complexifying the energy generates \mathcal{DS} with all the “nice” properties such dynamics entails. This leads to a dynamical point of view of *Kotani theory*, which expressed from the point of view of Theorem 1.1, studies the limiting properties of the invariant sections as $\text{Im } E \rightarrow 0+$. We mention that such a dynamical reformulation of Kotani theory has played an important roll in a dynamical description of the absolutely continuous spectrum of QPS [7, 6, 3, 4, 5], see also [8] for an even more general perspective (“monotonic cocycles”).

We structure the paper as follows. Sec. 2 briefly recalls the relation of the Jacobi cocycles (T, A^E) to solutions of the finite difference equations, $H_x \psi = E \psi$. We will also describe alternative choices for Jacobi cocycles appearing in the literature, for which Theorem 1.1 holds in an analogous form (see Theorem 2.1).

As mentioned earlier, the main point of this note is to account for *singular* Jacobi operators. For non-singular operators, Theorem 1.1 could be obtained by simple adaptations of the proof for Schrödinger operators, which we present in Sec. 3.

Sec. 4 forms the main part of the article, and is devoted to proving \mathcal{DS} outside the spectrum. The latter is done by verifying a *cone condition*. We outline the strategy in Sec. 4.1, the proof is carried out in Sec. 4.2. One noteworthy aspect of our proof is that it explicitly reveals the spectral theoretic meaning of the dynamical quantities involved. The invariant sections of the splitting are shown to be given in terms of the Weyl m -functions (m_{\pm}), with m_- giving rise to the dominating section, cf. (4.7). The key estimate, verifying the cone condition, is obtained in Proposition 4.1. It shows that the derivative of iterates of the Jacobi cocycle along the dominating section decays exponentially in the number of iterates, with a decay rate given by the Lyapunov exponent of the QPJ. Moreover, the angle between the invariant sections of the splitting is shown to be determined by the inverse of the Green's function of the QPJ, cf. (4.29)-(4.30).

We conclude with Sec. 5, where we show that \mathcal{DS} cannot occur on the spectrum.

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2. JACOBI COCYCLES

Fixing the spectral parameter $E \in \mathbb{C}$, solving $H_x \psi = E \psi$ over $\mathbb{C}^{\mathbb{Z}}$ can be formulated as iteration of the *measurable* cocycle¹ (T, B^E) applied to a given initial condition $\begin{pmatrix} \psi_0 \\ \psi_{-1} \end{pmatrix}$ for ψ ,

$$(2.1) \quad B_n^E(x) \begin{pmatrix} \psi_0 \\ \psi_{-1} \end{pmatrix} = \begin{pmatrix} \psi_n \\ \psi_{n-1} \end{pmatrix},$$

where $B^E(x) := \frac{1}{c(x)} A^E(x)$ and A^E is given in (1.4). This iterative procedure is a consequence of the second order difference nature of Jacobi operators. In spectral theory, it is better known as transfer matrix formalism, where the *transfer matrix* is given by $B^E(x)$.

Notice that since $\log |c| \in L^1(X, d\mu)$, the set $\mathcal{Z}(c) := \{x \in X : c(x) = 0\}$ is necessarily of μ -measure zero. In particular, positivity of μ on open sets, implies that (T, B^E) is well-defined and invertible on the full measure, and therefore *dense* G_{δ} -set,

$$(2.2) \quad X_0 := X \setminus \left(\bigcup_{n \in \mathbb{Z}} T^n \mathcal{Z}(c) \right).$$

¹One can weaken the definition of a (continuous) cocycle, requiring the matrix valued function $D : X \rightarrow M_2(\mathbb{C})$ to only be measurable with $\log_+ \|D(\cdot)\| \in L^1(X, d\mu)$, in which case (T, D) is called a *measurable cocycle*.

As $B^E(x)$ is only defined for μ -a.e. x , it is often more convenient to work with (T, A^E) , which inherits the continuity of the sampling functions c, v . We reiterate that presence of zeros in $c(x)$ translates to singularity of (T, A^E) .

An alternative choice for the transfer matrix B^E is given by,

$$(2.3) \quad \tilde{B}^E(x) = \frac{1}{c(T^{-1}x)} \begin{pmatrix} E - v(x) & -|c(T^{-1}x)|^2 \\ 1 & 0 \end{pmatrix},$$

which induces a complex symplectic, measurable cocycle, in particular

$$(2.4) \quad |\det \tilde{B}^E(x)| = 1, \mu\text{-a.e.}$$

Thus, for *non-singular* QPJ where $(T, \tilde{B}^E(x))$ is *continuous*, dynamical considerations reduce directly to the more familiar notion of uniform hyperbolicity. The latter is explored in Sec. 3.

The definition of $\tilde{B}^E(x)$ is suggested by the “scaled” discrete Laplacian in (1.1) [22, 12]; more specifically, $\psi \in \mathbb{C}^{\mathbb{Z}}$ satisfies $H_x\psi = E\psi$ if and only if

$$(2.5) \quad \tilde{B}_n^E(x) \begin{pmatrix} c(T^{-1}x)\psi_0 \\ \psi_{-1} \end{pmatrix} = \begin{pmatrix} c(T^{n-1}x)\psi_n \\ \psi_{n-1} \end{pmatrix}.$$

We mention that \tilde{B}^E is particularly natural in view of the Weyl m-function $m_-(x, E)$, cf (4.8) ².

For *singular* QPJ, however, as $\tilde{B}^E(x)$ is only defined for μ -a.e. x , one introduces in analogy to A^E above,

$$(2.6) \quad \tilde{A}^E(x) := \begin{pmatrix} E - v(x) & -|c(T^{-1}x)|^2 \\ 1 & 0 \end{pmatrix},$$

which induces the (continuous) cocycle derived from (T, \tilde{B}^E) . (T, \tilde{A}^E) thus yields an alternative Jacobi cocycle, which, like (T, A^E) , is singular precisely if c exhibits zeros.

Referring to (2.5), observe that the dynamics of the cocycles (T, A^E) and (T, \tilde{A}^E) is related by the measurable conjugacy³,

$$(2.7) \quad M(Tx)^{-1} \tilde{A}^E(x) M(x) = A^E(x), \quad M(x) := \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{c(T^{-1}x)} \end{pmatrix},$$

²We mention that the transfer matrix proposed in [22, 12] is in fact adapted to the Weyl m-function $m_+(x, E)$, hence differs from (2.3). In view of proving presence of a dominated splitting, it is however more natural to adapt the cocycle to $m_-(x, E)$, as the latter will be shown to give rise to the dominating section (cf Sec. 4). We mention that all arguments in this note carry over to the cocycles considered in [22, 12], in particular Theorem 1.1 also applies to those cocycles.

³As usual, a *measurable conjugacy* is a conjugacy between measurable cocycles where the mediating coordinate change in (2.7) is measurable with $\log \|M(\cdot)\| \in L^1(X, d\mu)$. The latter condition guarantees preservation of the LE. Note that log-integrability of the coordinate change under consideration in (2.7) follows from $\log |c| \in L^1(X, d\mu)$.

which in particular yields a relation between the top Lyapunov exponents,

$$(2.8) \quad L(T, A^E) = L(T, \tilde{A}^E), \quad L(T, B^E) = L(T, A^E) - \int \log |c| d\mu = L(T, \tilde{B}^E).$$

We mention that in spectral theory, $L(T, B^E) = L(T, \tilde{B}^E)$ is usually called *the* Lyapunov exponent of the QPJ.

For *non-singular* Jacobi operators, (2.7) becomes a *continuous* conjugacy, whence $(T, A^E) \in \mathcal{DS}$ if and only if $(T, \tilde{A}^E) \in \mathcal{DS}$.

In conclusion, we account for the flexibility in the choice of cocycles associated with the spectral theory of QPJ, formulating our main result also for (T, \tilde{A}^E) :

Theorem 2.1 (Theorem 1.1 ammended). *One has*

$$(2.9) \quad \Sigma = \{E : (T, A^E) \notin \mathcal{DS}\}.$$

The statement also holds when replacing (T, A^E) by (T, \tilde{A}^E) .

3. NON-SINGULAR JACOBI OPERATORS

As mentioned earlier, the spectral theory for *non-singular* Jacobi operators can be described by *continuous* cocycles with unimodular determinant similar to the Schrödinger case. In particular, since (T, A^E) and (T, \tilde{A}^E) are continuously conjugate, Theorem 1.1 is equivalently formulated as

$$(3.1) \quad \mathbb{C} \setminus \Sigma = \{E : (T, \tilde{B}^E) \in \mathcal{UH}\}.$$

Thus, Theorem 1.1 can be concluded from arguments along the lines of [19]. We briefly outline these straightforward adaptations.

To prove the “ \subseteq ” statement in (3.1), let $E \in \mathbb{C} \setminus \Sigma$ be given. It is well known that [24, 25, 26, 28] (see also, [31] for a more recent proof) a (continuous) cocycle (T, D) is *not* \mathcal{UH} if and only if for *some* $x_0 \in X$ and $v \in \mathbb{C}^2 \setminus \{0\}$,

$$(3.2) \quad \sup_{n \in \mathbb{Z}} \|D_n(x_0)v\| < \infty.$$

Thus, using (2.5), $(T, \tilde{B}^E) \notin \mathcal{UH}$ would imply that for some $x_0 \in X$, $H_{x_0}\psi = E\psi$ admits a bounded solution over $\mathbb{C}^{\mathbb{Z}}$, whence (see also 5.6) $E \in \Sigma$ - a contradiction.

In view of the “ \supseteq ” statement in (3.1), $(T, \tilde{B}^E) \in \mathcal{UH}$ clearly implies that all non-trivial solutions of $H_x\psi = E\psi$ over $\mathbb{C}^{\mathbb{Z}}$ increase exponentially on at least one of \mathbb{Z}_{\pm} . Thus, the Sch’nol-Berezanskii theorem [27, 11] (see (5.6)) and openness of \mathcal{UH} in the (continuous) cocycles with unimodular determinant implies $E \in \mathbb{C} \setminus \Sigma$ (cf Sec. 5).

4. DOMINATION OUTSIDE THE SPECTRUM

In this section we address the “ \supseteq ”-statement in (2.9). We start with some remarks on dominated splittings.

4.1. Dominated splittings and cone conditions. For $1 \leq j \leq 2$, let e_j denote the standard basis of \mathbb{C}^2 and set $E_j := \text{Span}\{e_j\}$. Taking advantage of the manifold structure of $\mathbb{P}\mathbb{C}^2$ induced by the charts,

$$(4.1) \quad \begin{aligned} \phi_1 : \mathbb{P}\mathbb{C}^2 \setminus E_1 &\rightarrow \mathbb{C} , \phi_1(\text{Span}\{(\begin{smallmatrix} v_1 \\ v_2 \end{smallmatrix})\}) = \frac{v_1}{v_2} , \\ \phi_2 : \mathbb{P}\mathbb{C}^2 \setminus E_2 &\rightarrow \mathbb{C} , \phi_2(\text{Span}\{(\begin{smallmatrix} v_1 \\ v_2 \end{smallmatrix})\}) = \frac{v_2}{v_1} , \end{aligned}$$

in local coordinates a given matrix $D = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq 0$ acts on $\mathbb{P}\mathbb{C}^2 \setminus \ker D$ as a linear fractional transformation.

We will denote the coordinate-free action of D on $\mathbb{P}\mathbb{C}^2 \setminus \ker D$ by $D \cdot z$ and its derivative, respectively, by $\partial D \cdot z$. Moreover, we will find it convenient to identify $\mathbb{P}\mathbb{C}^2$ with $\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ extending ϕ_2 in (4.1) to all of $\mathbb{P}\mathbb{C}^2$.

First, observe that a cocycle $(T, D) \in \mathcal{DS}$ if and only if *some* iterate is continuously conjugate to a diagonal cocycle, i.e. there exists $N \in \mathbb{N}$ and a coordinate change $M \in \mathcal{C}(X, GL(2, \mathbb{C}))$ such that

$$(4.2) \quad M(T^N x)^{-1} D_N(x) M(x) = \begin{pmatrix} \lambda_1(x) & 0 \\ 0 & \lambda_2(x) \end{pmatrix} ,$$

where $\lambda_j \in \mathcal{C}(X, \mathbb{C})$, $1 \leq j \leq 2$, satisfy

$$(4.3) \quad |\lambda_1(x)| > |\lambda_2(x)| , \text{ all } x \in X .$$

In particular, a necessary condition for $(T, D) \in \mathcal{DS}$ is a non-degenerate Lyapunov spectrum, i.e.

$$(4.4) \quad L(T, D) > \frac{1}{2} \int \log |\det D(x)| d\mu(x) .$$

A well-known technique to detect domination is to show existence of an *invariant cone field*. For $M_2(\mathbb{C})$ -cocycles this amounts to showing that for some $N \in \mathbb{N}$, there exists an open subset U of $\mathbb{P}\mathbb{C}^2$, $\overline{U} \subsetneq \mathbb{P}\mathbb{C}^2$ and uniformly transverse to $\ker D_N$, such that for all $x \in X$, $D_N(x) \cdot \overline{U} \subseteq U$. In this case, the dominating section $s_1(x)$ is determined iteratively by $s_1(x) = \cap_{k \in \mathbb{N}} D_{kN}(x) \cdot \overline{U}$.

Note also, that one necessarily has

$$(4.5) \quad \sup_{x \in X} |\partial D_N(x) \cdot s_1(x)| < 1 .$$

Conversely, suppose for some $N \in \mathbb{N}$ a given invariant section $s(\cdot) \in \mathcal{C}(X, \overline{\mathbb{C}})$ is uniformly transverse to $\ker D_N$ and satisfies the uniform contraction condition (4.5), then $(T, D) \in \mathcal{DS}$ with $s(x)$ defining the dominating section.

4.2. Proof of Theorem 2.1, “ \supseteq ”-statement. Set $\rho := \mathbb{C} \setminus \Sigma$. Throughout this section, let $E \in \rho$ be fixed. We will argue that a dominating, invariant section is provided from the spectral theory of Jacobi operators.

To this end, let $m_{\pm}(x, E)$ denote the standard *Weyl m -functions*,

$$(4.6) \quad m_{\pm}(x, E) := \langle \delta_{\pm 1}, (H_{x, \pm} - E)^{-1} \delta_{\pm 1} \rangle ,$$

defined in terms of the positive (negative) half-line operator $H_{x,\pm}$ acting on $l^2(\mathbb{Z}_\pm)$.

In view of (T, A^E) , we consider

$$(4.7) \quad \begin{aligned} s_-(x, E) &:= -c(T^{-1}x)m_-(x, E) , \\ s_+(x, E) &:= -\{\overline{c(T^{-1}x)}m_+(T^{-1}x, E)\}^{-1} , \end{aligned}$$

and for (T, \tilde{A}^E) , respectively,

$$(4.8) \quad \begin{aligned} \tilde{s}_-(x, E) &:= -m_-(x, E) , \\ \tilde{s}_+(x, E) &:= -\{|c(T^{-1}x)|^2m_+(T^{-1}x, E)\}^{-1} . \end{aligned}$$

Clearly, for all $E \in \mathbb{C} \setminus \mathbb{R}$,

$$(4.9) \quad 0 < |m_\pm(x, E)| < \frac{1}{|\operatorname{Im} E|} ,$$

thus $s_\pm(x, E), \tilde{s}_\pm(x, E)$ are well-defined with values in $\overline{\mathbb{C}}$.

To see that (4.7) and (4.8) are actually well defined for all $E \in \rho$, recall that H_x and $H_{x,-} \oplus H_{x,+}$ only differ by a finite rank perturbation, hence their essential spectra must agree. In particular, any real E in the resolvent set of H_x is either in the resolvent sets of both $H_{x,\pm}$ ⁴ or in the discrete spectrum⁵ of at least one of $H_{x,\pm}$.

Undefined expressions in (4.7)-(4.8) of the form “ $0 \times \infty$ ” are thus excluded; indeed, as $c(T^{-1}x) = 0$ implies $H_x = H_{x,-} \oplus H_{T^{-1}x,+}$, any $E \in \sigma_{\text{disc}}(H_{x,-}) \cup \sigma_{\text{disc}}(H_{T^{-1}x,+})$ would automatically be an eigenvalue of H_x .

We claim:

Lemma 4.1.

$$(4.10) \quad s_\pm(\cdot, \cdot), \tilde{s}_\pm(\cdot, \cdot) \in \mathcal{C}(X \times \rho, \overline{\mathbb{C}}) .$$

Proof. It suffices to show joint continuity of $m_\pm(x, E)$. Let $(x_0, E_0) \in X \times \rho$ be fixed and arbitrary. As outlined above, there are two possible situations to consider.

If E_0 is in the resolvent set of $H_{x_0,\pm}$, basic resolvent estimates (see e.g. Theorem 3.15 in [20]) imply that $(H_{x,\pm} - E)^{-1}$ exists and is jointly continuous in (x, E) (w.r.t operator norm for the x -dependence) in some open neighborhood of (x_0, E_0) ; in particular, $m_\pm(x, E)$ is jointly continuous at (x_0, E_0) .

Continuity for the case that E_0 is in the discrete spectrum of $H_{x_0,\pm}$ follows from a well-known fact on the stability of a finite system of isolated eigenvalues of finite multiplicity for a norm-continuous family of bounded operators (see e.g. Sec. IV.5 in [20]). \square

⁴In contrast to H_x , the resolvent sets for $H_{x,\pm}$ may in general depend on x .

⁵As usual, the *discrete spectrum* of a bounded operator is defined as the set of *isolated* eigenvalues of *finite* multiplicity; the remaining elements of the spectrum define the *essential spectrum*.

Recall, that the definition of the full-measure set $X_0 \subseteq X$ given in (2.2), guarantees that $c(T^k x) \neq 0$, $\forall k \in \mathbb{Z}$, whenever $x \in X_0$. For all $x \in X_0$, this in turn implies existence of solutions $\psi_{\pm}(x, E)$ of $H_x \psi = E\psi$ over $\mathbb{C}^{\mathbb{Z}}$ which are never zero, are l^2 at $\pm\infty$, and from (2.1) respectively (2.3), one has

$$(4.11) \quad s_{\pm}(x, E) = \frac{\psi_{\pm}(-1, x, E)}{\psi_{\pm}(0, x, E)}, \quad \tilde{s}_{\pm}(x, E) = \frac{\psi_{\pm}(-1, x, E)}{c(T^{-1}x)\psi_{\pm}(0, x, E)}, \quad x \in X_0,$$

and

$$(4.12) \quad -m_{-}(Tx, E)^{-1} = \begin{cases} (E - v(x)) - \overline{c(T^{-1}x)}s_{-}(x, E), \\ (E - v(x)) - |c(T^{-1}x)|^2\tilde{s}_{-}(x, E). \end{cases}$$

Moreover, uniqueness of the fundamental solutions $\psi_{\pm}(x, E)$ up to scalar multiples implies that for all $x \in X_0$,

$$(4.13) \quad A^E(x) \cdot s_{\pm}(x, E) = s_{\pm}(Tx, E), \quad \tilde{A}^E(x) \cdot \tilde{s}_{\pm}(x, E) = \tilde{s}_{\pm}(Tx, E).$$

We emphasize that outside X_0 above fundamental solutions $\psi_{\pm}(x, E)$ do not exist; indeed, $c(T^k x) = 0$ for some $k \in \mathbb{Z}$, implies decoupling of H_x whence any solution ψ of $H_x \psi = E\psi$ which is l^2 at $\pm\infty$ has to vanish *identically* in a neighborhood of $\pm\infty$ otherwise E is an eigenvalue of H_x .

In order to extend (4.12) and the invariance relation (4.13) to all of X , we use the following continuity arguments which also form the reason for requiring μ to be strictly positive on non-empty open sets of X .

First, since X_0 is dense in X , continuity of both sides of (4.12) as $\overline{\mathbb{C}}$ -valued functions implies that (4.12) extends to all of X . Moreover, one has:

Lemma 4.2. *For all $E \in \rho$ and all $x \in X$, $s_{-}(x, E)$ is transverse to $\ker A^E(x)$, similarly $\tilde{s}_{-}(x, E)$ is transverse to $\ker \tilde{A}^E(x)$. Moreover, for all $x \in X$*

$$(4.14) \quad A^E(x) \cdot s_{-}(x, E) = s_{-}(Tx, E), \quad \tilde{A}^E(x) \cdot \tilde{s}_{-}(x, E) = \tilde{s}_{-}(Tx, E).$$

Proof. We will focus on (T, A^E) , the argument for (T, \tilde{A}^E) being similar.

First observe that for $E \in \rho$, $A^E(x) \not\equiv 0$ since $c(x) = c(T^{-1}x) = 0$ would require $E - v(x) \neq 0$ for E to be in the resolvent set of H_x . In particular, $\dim \ker A^E(x) \leq 1$.

Based on (1.4), $\ker A^E(x)$ is non-trivial if one of the following two situations applies:

If $c(T^{-1}x) = 0$, $s_{-}(x, E) = 0$ which is automatically transverse to $\ker A^E(x) = E_2 \simeq \infty$.

If $c(x) = 0$, we may assume $c(T^{-1}x) \neq 0$, otherwise consider above. Then, $s_{-}(x, E) \simeq \text{Span}(\frac{1}{s_{-}(x, E)}) = \ker A^E(x)$ if and only if

$$(4.15) \quad (E - v(x)) - \overline{c(T^{-1}x)}s_{-}(x, E) = 0,$$

which as (4.12) holds on all of X would imply a pole of $m_{-}(Tx, E)$, or equivalently $E \in \sigma_{\text{disc}}(H_{Tx, -})$. The latter however is impossible for any $E \in \rho$ as $c(x) = 0$ yields $H_x = H_{Tx, -} \oplus H_{x, +}$.

Finally, uniform transversality of $s_-(\cdot, E)$ to $\ker A^E(\cdot)$ implies that $A^E(\cdot) \cdot s_-(\cdot, E) \in \mathcal{C}(X, \overline{\mathbb{C}})$, whence density of X_0 allows to extend (4.13) to all of X . \square

Based on the discussion in Sec. 4.1, we show that $(T, A^E), (T, \tilde{A}^E) \in \mathcal{DS}$ verifying a cone condition of the form (4.5). Recall that by the Combes-Thomas estimate (see e.g. [30], Lemma 2.5, for a formulation for Jacobi operators),

$$(4.16) \quad L(T, B^E) \geq \kappa \cdot \text{dist}(E; \Sigma) > 0, \quad E \in \rho,$$

where, uniformly over any compact neighborhood of Σ , $\kappa > 0$ can be chosen to only depend on $\|c\|_\infty$. In particular, using (4.4) and (2.8), (4.16) shows that for any $E \in \rho$ both $(T, A^E), (T, \tilde{A}^E)$ have a non-degenerate Lyapunov spectrum.

In view of the following, we let

$$(4.17) \quad \rho_- := \mathbb{C} \setminus (\Sigma \cup (\cup_{x \in X} \sigma_{\text{disc}}(H_{x, -}))) .$$

Clearly, $\mathbb{C} \setminus \mathbb{R} \subseteq \rho_-$, moreover $E \in \rho_-$ if $|E| \geq 2\|c\|_\infty + \|v\|_\infty$.

Proposition 4.1. *For any $E \in \rho_-$ one has*

$$(4.18) \quad \sup_{x \in X} |\partial A_n^E(x) \cdot s_-(x, E)| \leq \|c\|_\infty^2 e^{-2(L(T, A^E) + o(1))} e^{-2(n-1)(L(T, B^E) + o(1))} ,$$

as $n \rightarrow +\infty$. An analogous estimate holds for $\tilde{A}^E(x)$.

Remark 4.3. The proof also shows that the upper bound in (4.18) is optimal. Thus, as $L(T, A^E) \geq \int \log |c| d\mu$, N in the definition of \mathcal{DS} (4.4) will diverge whenever $L(T, B^E) \rightarrow 0$ as $\text{dist}(E; \Sigma) \rightarrow 0+$.

Proof. For brevity, we will focus on the cocycle (T, A^E) . First observe that for all $x \in X$, $s_-(x, E), m_-(Tx, E) \neq \infty$ as $E \in \rho_-$, whence from (4.12) we conclude

$$(4.19) \quad A^E(x) \cdot z = (\phi_2 \circ A^E \circ \phi_2^{-1})(z) = \frac{c(x)}{(E - v(x)) - c(T^{-1}x)z} ,$$

locally about $z = s_-(x, E)$ for all $x \in X$.

Thus, again using (4.12), we compute

$$(4.20) \quad \partial A^E(x) \cdot s_-(x, E) = \frac{c(x) \overline{c(T^{-1}x)}}{\left((E - v(x)) - \overline{c(T^{-1}x)} s_-(x, E) \right)^2}$$

$$(4.21) \quad = c(x) \overline{c(T^{-1}x)} m_-^2(Tx, E) .$$

From (4.13), (4.20), the chain rule implies

$$(4.22) \quad \begin{aligned} \partial A_n^E(x) \cdot s_-(x, E) &= \prod_{j=0}^{n-1} \partial A^E(T^j x) \cdot (A_j^E(x) \cdot s_-(x, E)) \\ &= \left(\prod_{j=0}^{n-1} c(T^j x) \right) \left(\prod_{j=-1}^{n-2} \overline{c(T^j x)} \right) \left(\prod_{j=1}^n m_-^2(T^j x, E) \right) , \quad n \geq 2 . \end{aligned}$$

Relating $m_-(x, E)$ to the fundamental solution $\psi_-(x, E)$, one determines, making use of ergodicity (see e.g. [30], Eq. (5.40) therein), that

$$(4.23) \quad \lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log |m_-(T^j x, E)| = -L(T, A^E), \text{ a.e. } x \in X.$$

Observe that since $m_-(x, E)$ is continuous with

$$(4.24) \quad 0 \leq |m_-(x, E)| < \infty, \quad E \in \rho_-,$$

unique ergodicity of T implies uniformity of the *upper* limit in (4.23) [15]⁶

$$(4.25) \quad \frac{1}{n} \sum_{j=0}^{n-1} \log |m_-(T^j x, E)| \leq -L(T, A^E) + o(1), \text{ uniformly for } x \in X.$$

The same argument yields

$$(4.26) \quad \frac{1}{n} \sum_{j=0}^{n-1} \log |c(T^j x)| \leq \int \log |c(x)| d\mu(x) + o(1),$$

uniformly in $x \in X$ as $n \rightarrow +\infty$.

Thus, combination of (2.8), (4.25), (4.26), and (4.22) yields (4.18) as claimed. \square

Thus, letting $S_{\pm}(\cdot, E)$ be continuous lifts of $s_{\pm}(\cdot, E)$ to the subspaces of \mathbb{C}^2 , Proposition 4.1 and (4.13) implies a \mathcal{DS} for all $E \in \rho_-$, with $S_-(x, E)$ the dominating- and $S_+(x, E)$ the minorating section; in particular for all $E \in \rho_-$ and $x \in X$,

$$(4.27) \quad A^E(x)S_+(x, E) \subseteq S_+(x, E),$$

$$(4.28) \quad A^E(x)S_-(x, E) = S_-(x, E).$$

Proposition 4.1 was restricted to $E \in \rho_-$ as this guaranteed boundedness of $m_-(x, E)$, which was crucial to conclude uniformity of the upper limit in (4.25). Nonetheless, having shown $(T, A^E), (T, \tilde{A}^E) \in \mathcal{DS}$ for all $E \in \mathbb{C} \setminus \mathbb{R}$, however already implies the same for all of ρ ; we provide an argument for (T, A^E) :

First, observe that one has:

Lemma 4.4. *For all $E \in \rho$, $S_{\pm}(\cdot, E)$ are uniformly transverse.*

⁶We use Theorem 1 in [15] which guarantees uniform convergence of *upper* limits in Caesáro means for continuous, sub-additive processes on a compact Hausdorff space, equipped with a uniquely ergodic dynamical system. In fact, it is easily seen that the proof extends to sub-additive processes $\{f_n\}$ which are upper semi-continuous and satisfy $\sup_{x \in X} f_1(x) \leq M < \infty$ (*upper* bound only !), which implies (4.25) and (4.26). We mention that recently, Furman's result has been extended to even encompass certain discontinuous process [21].

Proof. For any $E \in \rho$ and $x \in X_0$, standard expressions for the Green's function of H_x in terms of $\psi_{\pm}(x, E)$ yield

$$\begin{aligned} \langle \delta_0, (H_x - E)^{-1} \delta_0 \rangle^{-1} &= c(x) \left(\frac{\psi_+(1, x, E)}{\psi_+(0, x, E)} - \frac{\psi_-(1, x, E)}{\psi_-(0, x, E)} \right) \\ (4.29) \quad &= \overline{c(T^{-1}x)} \left(\frac{\psi_-(-1, x, E)}{\psi_-(0, x, E)} - \frac{\psi_+(-1, x, E)}{\psi_+(0, x, E)} \right), \end{aligned}$$

whence

$$(4.30) \quad |s_+(x, E) - s_-(x, E)| \geq \frac{\text{dist}(E; \Sigma)}{\|c\|_{\infty}}, \text{ all } x \in X_0.$$

By (4.9), $s_+(x, E)$, $s_-(x, E)$ can never *both* equal ∞ for $E \in \mathbb{C} \setminus \mathbb{R}$, whence (4.30) extends to all $x \in X$ by continuity.

Given *real* $E \in \rho$, using (4.30) there exists $\eta > 0$ only depending on $\text{dist}(E; \Sigma)$ such that for all $\epsilon > 0$,

$$(4.31) \quad \inf_{x \in X} \angle \{S_+(x, E + i\epsilon), S_-(x, E + i\epsilon)\} \geq \eta,$$

which by continuity implies the claimed transversality taking $\epsilon \rightarrow 0+$. \square

From Lemma 4.2, (4.28) already holds for all $E \in \rho$. Moreover, using Lemma 4.1, $S_+(\cdot, E)$ extends continuously to all of $E \in \rho$ whence so does (4.27).

In summary, taking $M(x) \in M_2(\mathbb{C})$ with the first (second) column vector in the direction of $S_-(x, E)$ ($S_+(x, E)$), using Lemma 4.4 implies $M(x) \in GL(2, \mathbb{C})$ and

$$(4.32) \quad M(Tx)^{-1} A^E(x) M(x) = \begin{pmatrix} \lambda_1(x) & 0 \\ 0 & \lambda_2(x) \end{pmatrix},$$

where $\lambda_j \in \mathcal{C}(X, \mathbb{C})$, $1 \leq j \leq 2$. As $S_-(x, E)$ is uniformly transversal from $\ker A^E$, $\lambda_1(x) \neq 0$, whence in particular by continuity,

$$(4.33) \quad \inf_{x \in X} |\lambda_1(x)| > 0.$$

Finally, non-triviality of the Lyapunov spectrum of (T, A^E) for all $E \in \mathbb{C} \setminus \Sigma$, (4.32)-(4.33), and unique ergodicity of T implies that for some $N \in \mathbb{N}$,

$$(4.34) \quad \left| \prod_{j=0}^N \lambda_1(x) \right| > \left| \prod_{j=0}^N \lambda_2(x) \right|, \text{ all } x \in X,$$

whence $(T, A^E) \in \mathcal{DS}$ (use (4.2) - (4.3)).

5. FINISHING UP ...

To complete the proof of Theorem 2.1, it is left to show that dominated splittings cannot occur on the spectrum. We will give an argument for (T, A^E) and comment on the simple adaptations for (T, \tilde{A}^E) in the end.

Suppose, for some $E \in \mathbb{C}$, $(T, A^E) \in \mathcal{DS}$, correspondingly giving rise to a conjugacy of the form (4.2). In particular,

$$(5.1) \quad \det A^E(x) = \frac{\det M(T^N x)}{\det M(x)} \lambda_1(x) \lambda_2(x) .$$

Proposition 5.1. *There is $\gamma > 0$ such that for all $x \in X_0$, there exists a subsequence $(k_l)_{l \in \mathbb{N}}$ of one of \mathbb{Z}_\pm such that for all $v \in \mathbb{C}^2 \setminus \{0\}$*

$$(5.2) \quad \|B_{k_l N}^E(x)v\| \gtrsim e^{k_l \gamma} , \text{ all } l \in \mathbb{N} .$$

Proof. Consider,

$$(5.3) \quad (A^E)^\sharp(x) := \frac{A^E(x)}{\sqrt{|\det A^E(x)|}} ,$$

i.e. on $X_0 \times \mathbb{C}^2$, $(\alpha, (A^E)^\sharp)$ is well-defined and invertible for all iterates $n \in \mathbb{Z}$ with $|\det(A^E)^\sharp(x)| = 1$.

Fix $x \in X_0$. Given $v \in \mathbb{C}^2 \setminus \{0\}$, decompose $v = v_1 + v_2$ with $v_j \in E_x^{(j)}$. Possibly changing to inverse dynamics, we may assume v_1 non-zero. From (5.1), we conclude for $k \in \mathbb{N}$

$$(5.4) \quad \|(A_{kN}^E)^\sharp(x)v_1\| \gtrsim \left| \frac{\det M(x)}{\det M(T^{kN}x)} \right|^{1/2} \|v_1\| \prod_{j=0}^{k-1} \left| \frac{\lambda_1(T^{jN}x)}{\lambda_2(T^{jN}x)} \right|^{1/2} \gtrsim \|v_1\| e^{\frac{1}{2}k \log \lambda} ,$$

where $\lambda = \inf_{x \in \mathbb{T}} \frac{|\lambda_1(x)|}{|\lambda_2(x)|} > 1$. Hence all non-trivial iterates of $(\alpha, (A^E)^\sharp)$ increase exponentially along a subsequence.

To conclude the same for solutions of $H_x \psi = E \psi$, observe that for $v \in \mathbb{C}^2$ and $n \in \mathbb{N}$,

$$(5.5) \quad \|(A_n^E)^\sharp(x)v\| = \frac{|c(T^n x)|^{1/2}}{|c(T^{-1}x)|^{1/2}} \|B_n^E(x)v\| ,$$

hence, by minimality, selecting a subsequence such that the scalar factor on the right hand side stays bounded away from zero, the claim follows. \square

For fixed $x \in X$, denote by $\mathcal{E}_g(H_x)$ the set of generalized eigenvalues of H_x , i.e. all $E \in \mathbb{C}$ which admit a polynomially bounded solution of $H_x \psi = E \psi$ over $\mathbb{C}^\mathbb{Z}$. From the theorem of Sch'nol-Berezanskii [27, 11] it is well-known that

$$(5.6) \quad \overline{\mathcal{E}_g(H_x)} = \Sigma ,$$

for all $x \in X$.

Claim 5.1 shows that for all $x \in X_0$, one has $E \notin \mathcal{E}_g(H_x)$. But then, since \mathcal{DS} is an open condition in $\mathcal{C}(X, M_2(\mathbb{C}))$, we conclude that for all $x \in X_0$, E cannot be a limit point of $\mathcal{E}_g(H_x)$ either. In summary, (5.6) implies the “ \subseteq ”-statement of Theorem 1.1.

In order to prove $(T, \tilde{A}^E) \notin \mathcal{DS}$ for any $E \in \Sigma$, again using (4.2) one shows that for all $x \in X_0$, $(T, \tilde{A}^E) \in \mathcal{DS}$ would lead to exponential increase of $\|\tilde{B}_n^E(x)v\|$ on

one of \mathbb{Z}_\pm , for all $v \in \mathbb{C}^2 \setminus \{0\}$. But then (2.5) would imply the content of Claim 5.1, which allows to proceed as above.

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